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Correlation Functions and Boundary Conditions in the Ising Ferromagnet

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A number of new results on the Ising ferromagnet are obtained as a consequence of correlation inequalities. These results concern the monotonicity properties of the correlation functions, the study of equilibrium states for certain boundary conditions, and the uniqueness of the state in a semi-infinite lattice.

KEY WORDS: Ising models; correlation inequalities.

1. INTRODUCTION

Recent investigations on the Ising ferromagnet have significantly improved our understanding of the statistical structure of the state in the two-phase coexistence region.[1-3] In particular, one knows, from results due to Dobrushin[4] and van Beijeren,[5] that non-translation-invariant equilibrium states exist for the model of three (or higher) dimensions at low temperature. These states show a sharp interface between two regions of opposite magnetization. Under the same conditions the two-dimensional system does not have a sharp interface; Gallavotti[6] and Abraham and Reed[7,8] have shown that in two dimensions the interface is diffuse, the width of the interface being of the order of the square root of its length. This supports the belief that there are no non-translation-invariant equilibrium states in two dimensions. It is also believed that the states encountered in Refs. 4 and 5 are the only extremal noninvariant states in three dimensions.

The analysis of these conjectures was the motivation for the study on which this paper reports. Let us say first that the existence of non-translation-invariant equilibrium states in two dimensions has not been completely ruled out. However, strong evidence for their absence is obtained from our

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results. These questions are analyzed in Section 4. A number of other results on related problems in the Ising ferromagnet are proved by the methods developed in this analysis. They concern the monotonicity properties of the correlations, which are discussed in Section 3, and the unicity of the state in a semiinfinite two-dimensional lattice, proved in Section 4. Some other remarks and applications of these results are mentioned along the way and in Section 5. Applications of correlation inequalities are the basic ingredients in all the proofs of the paper.

2. DEFINITION AND NOTATIONS

We consider a simple cubic lattice in d dimensions with unit edges and a parallelepipedic box Λ in the lattice. A lattice site will be represented by its Cartesian coordinates

$$x = \{x^1, x^2, \ldots, x^d\} \tag{1}$$

which take integer values. Λ may then be regarded as a subset of $\mathbb{Z}^d$. At each lattice point there is a spin $\sigma_x = \pm 1$.

The energy of a spin configuration $\sigma = \{\sigma_x\}$ is given by

$$H_\Lambda(\sigma) = -J \sum_{\langle x,y\rangle \subset \Lambda} \sigma_x \sigma_y - \sum_{x \in \Lambda} h_x \sigma_x \tag{2}$$

with associated probability measure

$$W_\Lambda(\sigma) = Z^{-1} \exp[-\beta H_\Lambda(\sigma)] \tag{3}$$

The first sum in (2) runs over nearest neighbor pairs in Λ and the second sum can be considered to represent an external magnetic field and/or a boundary term. In this last case one takes

$$h_x = J b_x, \qquad b_x = \pm 1 \tag{4}$$

for x in the boundary of Λ. J is assumed to be positive (ferromagnetic interaction) and $\beta = 1/kT$ denotes the inverse temperature.

Let σ_A be defined for any finite set A of lattice sites by

$$\sigma_A = \prod_{x \in A} \sigma_x \tag{5}$$

and let its expectation with respect to (3) be denoted by $\langle \sigma_A \rangle_\Lambda$. Then, an equilibrium or a Gibbs state of the system is the collection of all limits

$$\langle \sigma_A \rangle = \lim_{\Lambda \to \infty} \langle \sigma_A \rangle_\Lambda \tag{6}$$

for any convergent sequence of boxes increasing to $\mathbb{Z}^d$ with some boundary conditions.

The state $\langle\sigma_A\rangle$ is said to correspond to free boundary conditions if no boundary term is taken in (2). We let $\langle\sigma_A\rangle_{\Lambda(+)}$ and $\langle\sigma_A\rangle_{\Lambda(-)}$ denote the states that take in the boundary of Λ and all $b_x = +1$ and all $b_x = -1$, respectively. The limits (6) exist for such boundary terms and will be denoted $\langle\sigma_A\rangle^f$, $\langle\sigma_A\rangle^+$, and $\langle\sigma_A\rangle^-$. Examples of other boundary terms will be given in the following.

3. MONOTONICITY PROPERTIES OF THE CORRELATION FUNCTIONS

The inequality

$$\langle\sigma_{0,0}\sigma_{m,n}\rangle \geqslant \langle\sigma_{0,0}\sigma_{m,n+1}\rangle \qquad \text{if} \quad m \geqslant 0, \quad n \geqslant 0 \tag{7}$$

showing the decrease with the distance of the two-point correlation function, can be found in the book by McCoy and Wu (Ref. 9, p. 306) and in the references quoted in this chapter. It is given there as a conjecture coming from exact computations in two dimensions and zero magnetic field, where (7) is found in the particular case $m = 0$. In fact, as we shall see, this inequality holds in much more general situations as a consequence of Lebowitz inequalities. After having remarked this fact, we found out that Schrader[10] has obtained a proof of inequality (7) by using field-theoretical techniques. The method we use gives in fact a series of inequalities among which (7) is a particular case. They are summarized in the following theorem.

Theorem 1. Let us consider the Ising ferromagnet in d dimensions with a nonnegative, uniform magnetic field. A set A of lattice sites being given, let $\bar{A}$ and A^* denote the symmetric sets with respect to the planes $x^1 = -\frac{1}{2}$, $x^2 = -\frac{1}{2}$. If A and B are two finite sets of sites lying above these two planes, the following inequalities hold:

$$\langle\sigma_A\sigma_B\rangle \geqslant \langle\sigma_A\sigma_{\bar{B}}\rangle \tag{8}$$

$$\langle\sigma_A\sigma_B\rangle - \langle\sigma_A\sigma_{\bar{B}}\rangle \geqslant \langle\sigma_A\sigma_{B^*}\rangle - \langle\sigma_A\sigma_{\bar{B}^*}\rangle \tag{9}$$

The same inequalities hold for the symmetry with respect to the two diagonal planes $x^1 - x^2 = 0$ and $x^1 + x^2 = 0$ provided that both sets A and B lie strictly above these two planes. When the external field is zero, the correlations are assumed to correspond to the state with free or with $(+)$ boundary conditions.

To make more explicit the significance of these inequalities, we next write them in the particular case of the two-point correlation functions.

Corollary 1. In particular, we have

$$\langle\sigma_{00}\sigma_{mn}\rangle - \langle\sigma_{00}\sigma_{m,n+1}\rangle \geqslant \langle\sigma_{00}\sigma_{m+1,n}\rangle - \langle\sigma_{00}\sigma_{m+1,n+1}\rangle \geqslant 0, \quad \text{if} \quad m, n \geqslant 0 \tag{10}$$

$$\langle\sigma_{00}\sigma_{mn}\rangle - \langle\sigma_{00}\sigma_{m+1,n+1}\rangle \geqslant \langle\sigma_{00}\sigma_{m+1,n+1}\rangle - \langle\sigma_{00}\sigma_{m+2,n}\rangle \geqslant 0, \quad \text{if} \quad -m < n < m \tag{11}$$

where the indices indicate the first two coordinates in the lattice, the other coordinates being equal. (See also Figs. 1 and 2 for notations.)

Inequality (8) comes from Lebowitz inequalities applied to the duplicated spin variables introduced in Ref. 5, (9) from a next duplication and Ellis–Monroe inequalities.

Let $\bar{x} = \{-(x^1 + 1), x^2, ..., x^d\}$ be the reflection of the lattice site x with respect to the plane $x^1 = -\frac{1}{2}$. We assume that Λ is a symmetric parallelepipedic box and we denote by Λ_1 the set of points such that $x^1 \geqslant 0$ (the upper half of Λ); for all $x \in \Lambda_1$ we introduce the variables

$$s_x = \tfrac{1}{2}(\sigma_x + \sigma_{\bar{x}}); \qquad t_x = \tfrac{1}{2}(\sigma_x - \sigma_{\bar{x}}) \tag{12}$$

and the fields

$$H_x = h_x - h_{\bar{x}}; \qquad K_x = h_x - h_{\bar{x}} \tag{13}$$

Then, the Ising Hamiltonian can be written as

$$-H_\Lambda(\sigma) = 2J \sum_{\langle x,y\rangle \subset \Lambda_1} (s_x s_y + t_x t_y) + J \sum_{x\in\Lambda_1, x^1=0} (2s_x^2 + 1) + \sum_{x\in\Lambda_1} (H_x s_x + K_x t_x) \tag{14}$$

Under the conditions of the theorem $\{H_x \geqslant 0, K_x \geqslant 0\}$ the first Lebowitz inequality[11] applies to this Hamiltonian. Therefore

$$\langle s_A t_B\rangle \geqslant 0 \tag{15}$$

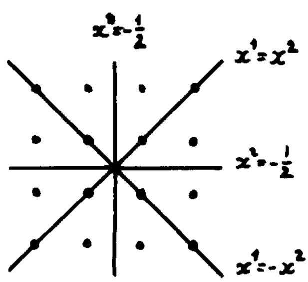

Fig. 1. Symmetry planes of the duplicated variables.

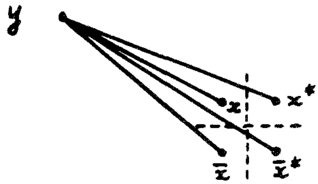

Fig. 2. The two-point correlation functions appearing in Theorem 1.

We continue to use in (15), and we shall also use in the following, the notation (6) for the products of the new site variables introduced into the discussion.

We remark that $(\sigma_A - \sigma_{\bar{A}})$ for $A \subset \Lambda_1$ is a sum of products of the s and t variables with positive coefficients; therefore

$$\langle(\sigma_A - \sigma_{\bar{A}})(\sigma_B - \sigma_{\bar{B}})\rangle \geqslant 0 \qquad \text{if} \quad A, B \subset \Lambda_1 \tag{16}$$

From (16) and the symmetry of Λ and $H_\Lambda(\sigma)$, inequality (8) follows.

Next we introduce a second symmetry in the lattice $x \to x^*$, $x^* = \{x^1, -(x^2 + 1),..., x^d\}$. We assume that Λ is also symmetric under this transformation and we denote by Λ_{12} the set of points $x \in \Lambda$ such that $x^1 \geqslant 0$ and $x^2 \geqslant 0$. For all $x \in \Lambda_{12}$ the following variables are introduced:

$$\begin{aligned} \alpha_x &= \tfrac{1}{2}(s_x + s_{x^*}); \qquad & \gamma_x &= \tfrac{1}{2}(t_{x^*} + t_x) \\ \beta_x &= \tfrac{1}{2}(s_x - s_{x^*}); \qquad & \delta_x &= \tfrac{1}{2}(t_{x^*} - t_x) \end{aligned} \tag{17}$$

It is easy to see that $H_\Lambda(\sigma)$ is again, under the conditions of the theorem, ferromagnetic in terms of these new variables, and that $\langle\sigma_{A^*} - \sigma_{\bar{A}^*} + \sigma_A - \sigma_{\bar{A}}\rangle$ for $A \subset \Lambda_{12}$ is a sum of products of these variables with positive coefficients. Hence, by applying the following Ellis–Monroe inequalities[9]

$$\langle\alpha_C \beta_D \gamma_E \delta_F\rangle \geqslant 0 \tag{18}$$

we get

$$\langle(\sigma_A - \sigma_{\bar{A}} - \sigma_{A^*} + \sigma_{\bar{A}^*})(\sigma_B - \sigma_{\bar{B}} - \sigma_{B^*} + \sigma_{\bar{B}^*})\rangle \geqslant 0 \quad \text{if} \quad A, B \subset \Lambda_{12} \tag{19}$$

which by the symmetry of Λ and H_Λ gives inequality (9).

Duplication with respect to the diagonal plane $x^1 = x^2$ is done in a similar way. A point x being given, its image will be $\bar{x} = \{x^2, x^1,..., x^d\}$. We let Λ' denote the set of all points x such that $x^1 > x^2$, in a symmetric box Λ. For $x \in \Lambda'$ the variables (12) and the fields (13) are introduced; for the points x such that $x^1 = x^2$ the original spin variables are conserved. The Ising Hamiltonian is then

$$-H_\Lambda(\sigma) = 2J \sum_{\langle x,y\rangle \subset \Lambda'} (s_x s_y + t_x t_y) + 2J \sum_{\substack{\langle x,y\rangle \subset \Lambda \\ x^1 = x^2; y \in \Lambda'}} \sigma_x s_y + \sum_{x \in \Lambda'} (H_x s_x + K_x t_x) + \sum_{x \in \Lambda, x^1 = x^2} h_x \sigma_x \tag{20}$$

By applying to this new Hamiltonian the same arguments as above, inequality (8) follows with the new interpretation. A second duplication with respect to the other diagonal plane can be done in a similar way and leads to a complete proof of Theorem 1.

The corollary corresponds to the particular case of Theorem 1 in which the sets A and B reduce to a point and the coordinate axes are chosen in such a way that B and its symmetric sites are nearest neighbors (Fig. 2).

A similar decreasing property of the correlations can also be deduced for the plane rotator model with nearest neighbor ferromagnetic interactions. In this model an angle variable θ_x is associated to each lattice point and the Hamiltonian is given by

$$-H_\Lambda = \sum_{\langle x,y\rangle \subset \Lambda} \{J_1 \cos\theta_x \cos\theta_y + J_2 \sin\theta_x \sin\theta_y\} + \sum_{x \in \Lambda} [h_1 \cos\theta_x + h_2 \sin\theta_x] \tag{21}$$

We shall assume that J_1, J_2, h_1, h_2 are positive constants. The duplicated variables can be introduced by

$$\alpha_x' = \tfrac{1}{2}(\cos\theta_x + \cos\theta_{\bar{x}}), \qquad \beta_x' = \tfrac{1}{2}(\cos\theta_x - \cos\theta_{\bar{x}})$$
$$\gamma_x' = \tfrac{1}{2}(\sin\theta_{\bar{x}} + \sin\theta_x) \qquad \delta_x' = \tfrac{1}{2}(\sin\theta_{\bar{x}} - \sin\theta_x)$$

and, similarly to the case of Ellis–Monroe inequalities, the positivity of the expectations of products of these variables can be obtained. The proof is easily done by adapting the arguments used in Ref. 13. From this fact the following proposition follows. Let A and B denote two sets of lattice sites lying above the plane $x^1 = -\frac{1}{2}$ and let $\bar{A}$ and $\bar{B}$ be the corresponding symmetric sets with respect to this plane; then we deduce the following corollary.

Corollary 2. For the Hamiltonian (21) the following inequalities hold:

$$\left\langle \prod_{x\in A} \cos\theta_x \prod_{x\in B} \cos\theta_x \right\rangle \geqslant \left\langle \prod_{x\in A} \cos\theta_x \prod_{x\in \bar{B}} \cos\theta_x \right\rangle \tag{22}$$

$$\left\langle \prod_{x\in A} \sin\theta_x \prod_{x\in B} \sin\theta_x \right\rangle \geqslant \left\langle \prod_{x\in A} \sin\theta_x \prod_{x\in \bar{B}} \sin\theta_x \right\rangle \tag{23}$$

$$\left\langle \prod_{x\in A} \sin\theta_x \prod_{x\in B} \cos\theta_x \right\rangle \leqslant \left\langle \prod_{x\in A} \sin\theta_x \prod_{x\in \bar{B}} \cos\theta_x \right\rangle \tag{24}$$

In particular, the inequality [corresponding to (7)]

$$\langle\cos(\theta_{00} - \theta_{mn})\rangle \geqslant \langle\cos(\theta_{00} - \theta_{m,n+1})\rangle \qquad \text{if} \quad m \geqslant 0, \quad n \geqslant 0$$

showing the decreasing of the two-point correlation functions, is obtained by adding the first two inequalities above in the case where A and B reduce to a point.

As an application of Theorem 1, we give the following proposition.

Corollary 3. Let us consider in the two-dimensional Ising ferromagnet the two-point correlation functions

$$f_\theta(L) = \langle\sigma_{00}\sigma_{mn}\rangle, \qquad L = (m^2 + n^2)^{1/2}, \qquad \tan\theta = n/m \tag{25}$$

at large distances, the direction θ being kept fixed. At the critical point

$$f_\theta(L) = O(L^{-1/4}) \qquad \text{for} \quad L \to \infty \tag{26}$$

At the critical point the behavior of $f_\theta(L)$ in the diagonal directions $\theta = \pm\frac{1}{4}\pi$ is known[9]; it decreases as $L^{-1/4}$. Away from the critical points, the truncated two-point function decreases exponentially when $L \to \infty$. From the second inequalities in (10) and (11) we can estimate $f_\theta(L)$ (see Fig. 3); then, for instance,

$$f_{\pi/4}(L_1) \leqslant f_\theta(L) \leqslant f_{\pi/4}(L_2); \qquad L_1 = L\cos(\tfrac{1}{2}\pi - \theta); \quad L_2 = 2^{1/2}L\cos\theta \tag{27}$$

in the case $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. From this, Corollary 2 follows.

Some more general Hamiltonians and boundary terms can be considered provided that they are ferromagnetic in the duplicated variables. For instance, inequality (7), which can be written exactly as $\langle t_x t_y\rangle \geqslant 0$, is a particular case of Lebowitz's $\langle t_A\rangle \geqslant 0$, and holds therefore if $K_x \geqslant 0$ for any H_x. As a second example we can take a ferromagnetic model with nearest and next nearest neighbor interactions. Duplication with respect to a diagonal plane can be done as before, leading to a Hamiltonian analogous to (19), and therefore inequalities (11) hold also in this case. Duplication

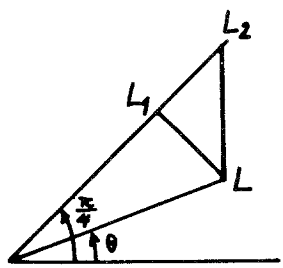

Fig. 3. Comparison of the two-point correlation functions at large distances in Corollary 3.

with respect to the planes orthogonal to the coordinate axis can also be considered, by keeping the variable σ_x for x in the plane $x^1 = 0$ and duplicating the spins symmetric with respect to the plane $x^1 = 0$ as in Ref. 5. One obtains $\langle\sigma_{00}\sigma_{m,n}\rangle \geqslant \langle\sigma_{00}\sigma_{m,n+1}\rangle$.

Extensions of Theorem 1 [inequality (8)] can also be obtained for higher order or continuous spins as well as in the $P(\varphi)$ field theory models. The inequality (9) of Theorem 1 holds in systems of continuous spins and φ^4 models, which reduce in fact to spin-$\frac{1}{2}$ systems; we refer for example to Refs. 13 and 14 for the needed inequalities. Finally, let us mention that certain applications of these inequalities to the $P(\varphi)$ field theory model have been proposed in Ref. 10.

The remarks above will apply also to the results concerning estimations in the following sections.

4. ON EQUILIBRIUM STATES AND BOUNDARY CONDITIONS

In this section we prove that a large class of boundary conditions give translation-invariant equilibrium states in the two-dimensional Ising ferromagnet at zero magnetic field and therefore that no rigid interface is present in these cases. (For $h \neq 0$ the equilibrium state is known to be unique.)

The initial idea for the following treatment can be described as follows. Let us consider the difference

$$\langle\sigma_{m,n+1}\rangle - \langle\sigma_{m,n}\rangle$$

between the magnetizations at two nearest neighbor points and suppose that we want to make it as large as possible. It seems plausible that this should occur for the boundary condition corresponding to taking the spins

$$\Lambda(\pm): \quad b_{m',n'} = \pm 1 \quad \text{if} \quad n' \geqslant n+1; \qquad b_{m',n'} = -1 \quad \text{if} \quad n' \leqslant n$$

on the boundary, which are in fact the boundary conditions considered in Refs. 4–8. This intuition relies on the assumption that the spin $\sigma_{m,n+1}$ should be more affected by the $+1$ spins than by the -1 spins on the boundary, while for the spin σ_{mn} the contrary occurs, i.e., that the influence of the boundary on the state at a lattice point decreases with the distance, a fact more or less related to the decreasing property (7) of the correlations. If it could be proved that the difference between the two magnetizations takes its largest value for the condition $\Lambda(\pm)$, it would follow that the obtained state is in some sense extremal, and, by applying the results in Refs. 7 and 8, that the magnetization is invariant in two dimensions. This statement unfortunately has not been proved in general, but the argument above has suggested to us to apply the correlation inequalities in the treatment of the problems considered in this paper.

Another approach to these problems, using contour techniques at low temperature (as in Ref. 6), has been developed by Higuchi.[15] Also in this case a complete answer to the conjecture on the absence in two dimensions of non-translation-invariant states has not been found. Earlier results concerning the discussion in this section have been given in Ref. 16.

We shall first study the limiting state obtained from the boundary condition $\Lambda(\pm)$. As mentioned in the introduction, this state was first studied by Gallavotti,[6] who showed that it is translation invariant at low temperature. Abraham and Reed[7,8] showed later, by direct computations valid at any temperature, that the magnetization is zero in the middle of a big box. By using this last result we shall give here a complete proof of the translation invariance of this state.

Proposition 1. Let $\langle\sigma_A\rangle_{\Lambda(\pm)}$ denote the correlation functions for the Ising ferromagnet in a box Λ, symmetric with respect to the plane $x^1 = -\frac{1}{2}$, with the boundary term corresponding to $b_x = +1$ if $x^1 \geqslant 0$ and $b_x = -1$ if $x^1 \leqslant -1$. Let s_x and t_x be defined as in (12). Then the limits

$$\lim_{\Lambda\to\infty} \langle s_A\rangle_{\Lambda(\pm)} = \langle s_A\rangle^{\pm}; \qquad \lim \langle t_A\rangle_{\Lambda(\pm)} = \langle t_A\rangle^{\pm} \tag{28}$$

exist and define correlation functions which are invariant under the lattice translations orthogonal to the x^1 axis. Furthermore,

$$\lim_{|x|\to\infty; x^1=0} \langle t_{A+x} t_B\rangle^{\pm} = \langle t_A\rangle^{\pm}\langle t_B\rangle^{\pm} \tag{29}$$

for these translations.

We first notice that, since the Hamiltonian can be written in the form (15), the second Lebowitz inequalities[11]

$$\begin{aligned} \langle s_A s_B\rangle &\geqslant \langle s_A\rangle\langle s_B\rangle \\ \langle t_A t_B\rangle &\geqslant \langle t_A\rangle\langle t_B\rangle \\ \langle s_A t_B\rangle &\leqslant \langle s_A\rangle\langle t_B\rangle \end{aligned} \tag{30}$$

hold, provided that $H_x \geqslant 0$ and $K_x \geqslant 0$ for all $x \in \Lambda_1$. This shows that $\langle s_A\rangle$ decreases and $\langle t_A\rangle$ increases when the fields K_x increase. Hence $\langle s_A\rangle_{\Lambda(\pm)}$ and $\langle t_A\rangle_{\Lambda(\pm)}$ are monotone functions of the box Λ with respect to the inclusion. Then the same arguments, due to Griffiths (partially reported in Ref. 17) which have been used for the state $\langle\sigma_A\rangle_{\Lambda(+)}$ (see, for instance, Lemma 2.4 in Ref. 18) can be applied. From these arguments, the existence of the limits (28), the translation invariance, and the cluster property (29) follow.

In particular, since

$$\begin{aligned} \langle s_x s_y\rangle^{\pm} &= \tfrac{1}{2}\langle\sigma_x\sigma_y\rangle^{\pm} + \tfrac{1}{2}\langle\sigma_x\sigma_{\bar{y}}\rangle^{\pm} \\ \langle t_x t_y\rangle^{\pm} &= \tfrac{1}{2}\langle\sigma_x\sigma_y\rangle^{\pm} - \tfrac{1}{2}\langle\sigma_x\sigma_{\bar{y}}\rangle^{\pm} \end{aligned} \tag{31}$$

the two-point correlation functions $\langle\sigma_x\sigma_y\rangle^\pm$ exist for any pair (x, y) and are translation invariant in the directions orthogonal to the x^1 axis.

Proposition 2. Let us consider the Ising ferromagnet in two dimensions and let $\langle\sigma_A\rangle_{\Lambda(\pm)}$ be defined as in Proposition 1. Then, the limiting state is translation invariant and

$$\langle\sigma_A\rangle^\pm = \lim_{\Lambda\to\infty} \langle\sigma_A\rangle_{\Lambda(\pm)} = \tfrac{1}{2}\{\langle\sigma_A\rangle^+ + \langle\sigma_A\rangle^-\} \tag{32}$$

The proof is based on the Abraham and Reed result that $\langle\sigma_x\rangle^\pm = 0$.

Let $e_1 = \{1, 0\}$ be the unit vector in the x^1 direction, and introduce the occupation number variables

$$n_x = \tfrac{1}{2}(\sigma_x + 1) \tag{33}$$

Let us assume that

$$\begin{aligned} \Lambda &= \{x \in \mathbb{Z}^2;\quad -N \leqslant x^1 \leqslant N-1,\quad -N \leqslant x^2 \leqslant N-1\} \\ \Lambda' &= \{x \in \mathbb{Z}^2;\quad -(N+1) \leqslant x^1 \leqslant N-1,\quad -N \leqslant x^2 \leqslant N-1\} \end{aligned} \tag{34}$$

Then, by the FKG inequalities[19] we get

$$\langle n_A\rangle_{\Lambda(\pm)} \leqslant \langle n_A\rangle_{\Lambda'(\pm)} \leqslant \langle n_{A+e_1}\rangle_{\Lambda(\pm)} \tag{35}$$

We notice (Fig. 4) that one passes from the second to the first system by introducing a negative field $h_x \to -\infty$ for all $x \in \Lambda$ in the layer $x^1 = -(N+1)$, and from the second to the third system by introducing a positive field $h_x \to +\infty$ for all $x \in \Lambda$ in the layer $x^1 = N - 1$ and $a + 1$ boundary spin in the two points with $x^1 = -1$. Since $(\sum_{x\in A} n_x) - n_A$ is an increasing function in the n variables, the FKG inequalities imply also that

$$\left\langle \left(\sum_{x\in A} n_x\right) - n_A \right\rangle_{\Lambda(\pm)} \leqslant \left\langle \left(\sum_{x\in A} n_{x+e_1}\right) - n_{A+e_1} \right\rangle_{\Lambda(\pm)} \tag{36}$$

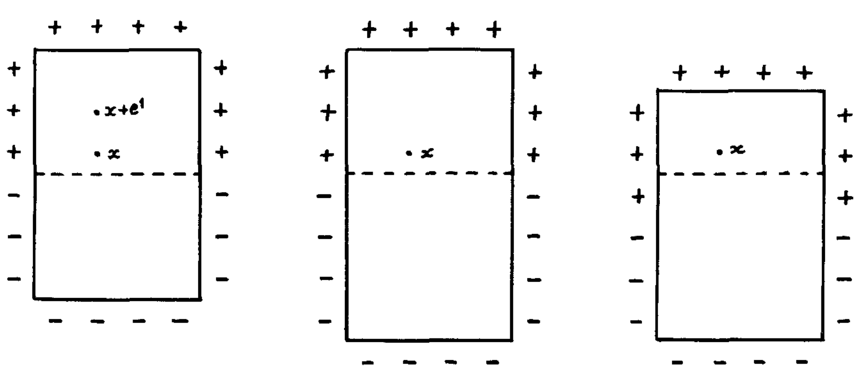

Fig. 4. Boundary terms appearing in the proof of Proposition 2.

Therefore

$$0 \leqslant \langle n_A \rangle_{\Lambda(\pm)} - \langle n_{A+e_1} \rangle_{\Lambda(\pm)} \leqslant \sum_{x \in A} (\langle n_x \rangle_{\Lambda(\pm)} - \langle n_{x+e_1} \rangle_{\Lambda(\pm)}) \tag{37}$$

But $\langle n_x \rangle_{\Lambda(\pm)} \to \frac{1}{2}$ when $\Lambda \to \infty$ for all x. Hence the state is translation invariant in the x^1 direction (the use of such an increasing function was introduced by Lebowitz and Martin-Löf[20]).

From this and Proposition 1 we deduce that the two-point correlation functions are translation invariant. Hence

$$\langle \sigma_x \sigma_y \rangle^{\pm} = \langle \sigma_x \sigma_y \rangle^{+} \tag{38}$$

from the fact, proved in Ref. 21, that the translation-invariant correlation functions with an even number of points are unique in the two-dimensional Ising ferromagnet. Although this fact is not explicitly stated, it follows easily from the proof of Theorem 1 in Ref. 21.

Now we apply a recently discovered inequality by Lebowitz.[22] In our case it says that

$$\langle \sigma_A \rangle_{\Lambda(+)} - \langle \sigma_A \rangle_{\Lambda} \geqslant |\langle \sigma_B \rangle_{\Lambda(+)} \langle \sigma_A \sigma_B \rangle_{\Lambda} - \langle \sigma_B \rangle_{\Lambda} \langle \sigma_A \sigma_B \rangle_{\Lambda(+)}| \tag{39}$$

where $\langle \sigma_A \rangle_{\Lambda}$ denotes the correlation functions associated with an arbitrary boundary term in the box Λ. From (39) it follows that if, when $\Lambda \to \infty$ the two-point correlations of both systems coincide, then all even correlation functions coincide. This allows us to deduce, by taking into account the results in Ref. 21, that then the state obtained from $\langle \sigma_A \rangle_{\Lambda}$ when $\Lambda \to \infty$ is translation invariant and therefore a linear convex combination of the $(+)$ and $(-)$ states.

In the present case, from (38) we get (32).

We shall next consider more general boundary conditions.

Lemma 1. Let us consider the Ising ferromagnet in d dimensions. Assume that the box Λ is symmetric with respect to the plane $x^1 = -\frac{1}{2}$ and introduce the notations (12) and (13). Let us denote by the superscripts H and K the original system, and by the superscripts $|H|$ and $|K|$ the systems in which H_x and K_x have been replaced by $|H_x|$ and $|K_x|$ for all $x \in \Lambda$. Then

$$\text{if} \quad H_x \geqslant 0, \qquad \langle s_A \rangle^{|K|} \leqslant \langle s_A \rangle^{K}, \qquad \langle t_A \rangle^{|K|} \geqslant \langle t_A \rangle^{K} \tag{40a}$$

$$\text{if} \quad K_x \geqslant 0, \qquad \langle s_A \rangle^{|H|} \geqslant \langle s_A \rangle^{H}; \qquad \langle t_A \rangle^{|H|} \leqslant \langle t_A \rangle^{H} \tag{40b}$$

We shall prove only (40a) since (40b) is derived in a similar way. Let us consider the Hamiltonian

$$\begin{aligned} -H_\Lambda = 2J \sum_{\langle x,y \rangle \subset \Lambda_1} (s_x s_y + t_x t_y) + J \sum_{x \in \Lambda_1;\, x^1 = 0} [2(s_x)^2 - 1] \\ + \sum_{x \in \Lambda_1} H_x s_x + \sum_{x \in \Lambda_1;\, K_x \geq 0} K_x t_x + \sum_{x \in \Lambda_1;\, K_x < 0} |K_x| t_x \sigma_\alpha \end{aligned} \tag{41}$$

in which an additional spin variable $\sigma_\alpha = \pm 1$ has been introduced. When $\sigma_\alpha = +1$, (41) becomes the Hamiltonian corresponding to the system $|K|$, and for $\sigma_\alpha = -1$, (41) becomes the Hamiltonian of the system K. The second Lebowitz inequality

$$\langle t_A \sigma_\alpha \rangle \geqslant \langle t_A \rangle \langle \sigma_\alpha \rangle \tag{42}$$

for the system (41) gives then

$$\frac{\langle t_A \rangle Z^{|K|} - \langle t_A \rangle^K Z^K}{Z^{|K|} - Z^K} \geqslant \frac{Z^{|K|} \langle t_A \rangle^{|K|} + Z^K \langle t_A \rangle^K}{Z^{|K|} + Z^K}$$

which after simplification becomes

$$\langle t_A \rangle^{|K|} \geqslant \langle t_A \rangle^K$$

In the same way the inequality

$$\langle s_A \sigma_\alpha \rangle \leqslant \langle s_A \rangle \langle \sigma_\alpha \rangle \tag{43}$$

for the system (41) proves the first inequality in (40a).

From the lemma the boundary conditions

$$B_0\colon\ b_x = +1,\quad b_{\bar{x}} = \pm 1; \qquad B\colon\ b_x + b_{\bar{x}} \geqslant 0; \qquad B'\colon\ b_x - b_{\bar{x}} \leqslant 0$$

can be compared. They correspond to the following possibilities in the symmetric boundary points:

$$B_0\colon \begin{Bmatrix} + & + \\ + & - \end{Bmatrix}; \qquad B\colon \begin{Bmatrix} + & + & - \\ + & - & + \end{Bmatrix}; \qquad B'\colon \begin{Bmatrix} + & - & + \\ + & - & - \end{Bmatrix} \tag{44}$$

Let us suppose that $b_{\bar{x}}$ coincides in the systems $\Lambda(B_0)$, $\Lambda(B)$, and $\Lambda(B')$; then

$$\langle s_A \rangle_{\Lambda(B)} \geqslant \langle s_A \rangle_{\Lambda(B_0)} \geqslant \langle s_A \rangle_{\Lambda(B')} \tag{45}$$

$$\langle t_A \rangle_{\Lambda(B)} \leqslant \langle t_A \rangle_{\Lambda(B_0)} \leqslant \langle t_A \rangle_{\Lambda(B')} \tag{46}$$

On the other hand, the Lebowitz inequalities yield

$$\langle s_A \rangle_{\Lambda(B_0)} \geqslant \langle s_A \rangle_{\Lambda(\pm)} \tag{47}$$

$$\langle t_A \rangle_{\Lambda(B_0)} \leqslant \langle t_A \rangle_{\Lambda(\pm)} \tag{48}$$

Theorem 2. Let us consider the two-dimensional Ising ferromagnet. Let $\Lambda(B)$ be a symmetric box with respect to the line $x^1 = -\frac{1}{2}$ with a boundary term satisfying the condition $b_x + b_{\bar{x}} \geqslant 0$ (or the condition $b_x + b_{\bar{x}} \leqslant 0$) for all pairs $(x, \bar{x})$ of symmetric boundary points. Then, the limiting state is translation invariant and therefore

$$\lim_{\Lambda \to \infty} \langle \sigma_A \rangle_{\Lambda(B)} = \langle \sigma_A \rangle^B = \lambda \langle \sigma_A \rangle^+ + (1 - \lambda) \langle \sigma_A \rangle^- \tag{49}$$

for a certain λ, $0 \leqslant \lambda \leqslant 1$.

We consider the case in which the condition $b_x + b_{\bar{x}} \geqslant 0$ is satisfied; the same statement follows in the case $b_x + b_{\bar{x}} \leqslant 0$ by the spin reversal symmetry. From inequalities (45) and (47) and Proposition 2 we get

$$\langle s_x s_y \rangle^B \geqslant \langle s_x s_y \rangle^{\pm} = \langle s_x s_y \rangle^{+} \tag{50}$$

Since

$$4 s_x s_y = \sigma_x \sigma_y + \sigma_{\bar{x}} \sigma_{\bar{y}} + \sigma_x \sigma_{\bar{y}} + \sigma_{\bar{x}} \sigma_y \tag{51}$$

and since for any pair of points $\langle \sigma_x \sigma_y \rangle^B \leqslant \langle \sigma_x \sigma_y \rangle^{+}$, it follows from (50) that

$$\langle \sigma_x \sigma_y \rangle^B = \langle \sigma_x \sigma_y \rangle^{+} \tag{52}$$

The two-point correlation functions of the B state coincide with those of the (+) state. Then the argument used in the last part of the proof of Proposition 2 can be applied, and then Theorem 2 follows.

Figure 5 shows examples of boundary conditions in which Theorem 2 applies.

We remark that from the Abraham and Reed result and the considerations above, it follows precisely that if $\langle \sigma_A \rangle_{\Lambda(B)}$ is the state of Theorem 2 in a large square box of side $L \to \infty$, then $\langle \sigma_A \rangle_{\Lambda(B)}$ is translation invariant in the region $|x^1| < L_\alpha$ for $0 \leqslant \alpha < \frac{1}{2}$.

The possibility of the existence of non-translation-invariant states in the two-dimensional Ising model looks very unlikely after Theorem 2. First, it eliminates a large class of boundary conditions. Second, we notice that the conditions on the boundary terms are referred to a particular horizontal line which does not play a role in the conclusion.

5. ON THE EQUILIBRIUM STATE IN A SEMIINFINITE LATTICE

In this section a proof is given of the unicity of the state in a semi-infinite lattice in two dimensions. This fact appears as a direct consequence

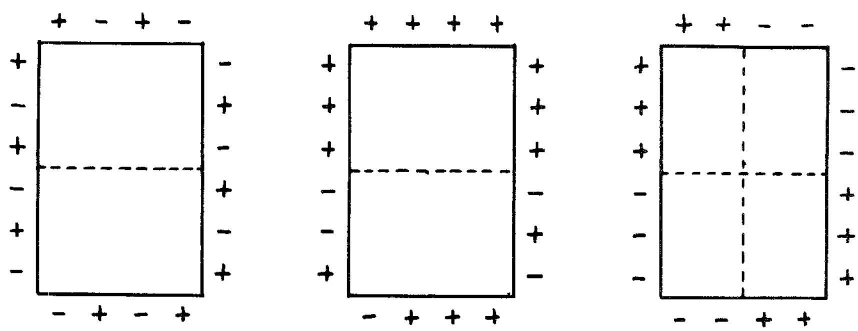

Fig. 5. Examples of boundary conditions in which Theorem 2 applies.

of the absence of a sharp interface in the $(\pm)$ state. A discussion of this problem has also been announced by Dobrushin in Ref. 23.

Theorem 3. Let us fix for all points in the layer $x^1 = 0$ the spins $\sigma_x = +1$. Let A be a finite set of points situated above this layer (i.e., such that $x^1 > 0$) and Q a rectangular box, its lower side coinciding with the line $x^1 = 0$ and arbitrary boundary conditions in the other sides. The semi-infinite state is defined as a limit of the form

$$\langle\sigma_A\rangle_{\text{s.i.}} = \lim_{Q\to\infty} \langle\sigma_A\rangle_Q \tag{53}$$

for some sequence of boxes increasing to the semiinfinite lattice $x' \geqslant 0$ with some boundary terms. In the two-dimensional Ising ferromagnet the semiinfinite state is unique.

Let x be a point above the layer $x^1 = 0$ and let $\langle t_x\rangle_{\Lambda(B_0)}$ be defined as in (44) and (46). In the system associated with $\Lambda(B_0)$ we introduce a positive field H_x for $x^1 = 0$, which increases to $+\infty$. We obtain then a system denoted $\Lambda(s)$ in which $\sigma_x = +1$ for all points in the two layers $x^1 = 0$ and $x^1 = -1$. From the Lebowitz inequalities (30) we get

$$\langle t_x\rangle_{\Lambda(B_0)} \geqslant \langle t_x\rangle_{\Lambda(s)} = \langle\sigma_x\rangle_{Q(+)} - \langle\sigma_x\rangle_Q \tag{54}$$

where Q denotes the semi-system with the same boundary term as in the lower half of the box $\Lambda(B_0)$. It can therefore be any boundary term (Fig. 6). From Theorem 2, it follows that

$$0 \leqslant \lim_{Q\to\infty} (\langle\sigma_x\rangle_{Q(+)} - \langle\sigma_x\rangle_Q) \leqslant \lim_{\Lambda\to\infty} \langle t_x\rangle_{\Lambda(B_0)} = 0 \tag{55}$$

As in the proof of Proposition 2, we now apply FKG inequalities. Then

$$\langle n_A\rangle_{Q(+)} \geqslant \langle n_A\rangle_Q, \qquad \left\langle \left(\sum_{x\in A} n_x\right) - n_A \right\rangle_{Q(+)} \geqslant \left\langle \left(\sum_{x\in A} n_x\right) - n_A \right\rangle_Q \tag{56}$$

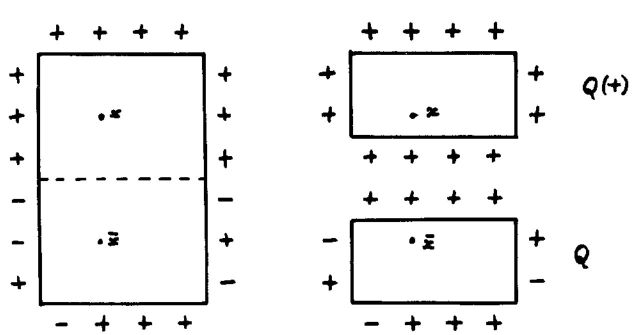

Fig. 6. Construction for the proof of Theorem 3.

From (55) and (56) we have

$$\lim_{Q\to\infty} \langle n_A \rangle_Q = \lim_{Q\to\infty} \langle n_A \rangle_{Q(+)} \tag{57}$$

which proves the unicity of the semiinfinite state.

We remark finally that Theorem 3 is a particular case of a more general statement which can be proved similarly. Instead of placing the $+1$ spins on the lines $x^1 = 0$, $x^1 = -1$, one can put them arbitrarily in symmetric points and the same consequences follow. This yields the following proposition.

Let an infinite connected line γ of $+1$ spins be given, entirely lying above any given horizontal line. Then the correlation functions for points above the line γ are unique. The same fact is true under the condition that all points of γ have ordinates bounded from below as the square root of its distance to the origin (exactly, from the Abraham and Reed results, if for all $x \in \gamma$, $x^1 > -(x^2)^\alpha$ for a certain α, $0 \leqslant \alpha \leqslant \frac{1}{2}$).

6. ON NON-TRANSLATION-INVARIANT EQUILIBRIUM STATES IN THREE OR MORE DIMENSIONS

We next make some brief comments concerning the Ising ferromagnet in higher dimensions. Clearly Theorems 2 and 3 still hold in three dimensions if no rigid interface is present for the $\Lambda(\pm)$ boundary condition. We recall that there is the belief[5] that this interface ceases to be rigid at a certain temperature T_R less than the critical temperature.

Other non-translation-invariant states different from the states $(\pm)$ exist in three dimensions. Take, for instance, in a cubic box Λ centered at the origin the boundary condition

$$b_x = +1 \text{ if } x^1 \geqslant 0 \text{ and if } x^1 \geqslant -1 \text{ and } x^2 \geqslant 0; \quad b_x = -1 \text{ otherwise} \tag{58}$$

Comparison by FKG inequalities with the $(\pm)$ state shows that the limiting state of (58) is noninvariant in the x^1 direction. It is thought, however, that it is in fact a linear convex combination of different $(\pm)$ states, which should be the only extremal noninvariant states existing in the model. In other words, this means that there can only be planes parallel to the faces of the lattice cubes at finite distance, and no angles, corners, or diagonal planes as rigid interfaces. These kinds of interfaces would correspond to the following boundary terms in the box:

$$b_x = +1 \quad \text{if} \quad x^1 \geqslant 0 \quad \text{and} \quad x^2 \geqslant 0; \qquad b_x = -1 \quad \text{otherwise} \tag{59}$$

$$b_x = +1 \quad \text{if} \quad x^1 \geqslant 0, \quad x^2 \geqslant 0, \quad \text{and} \quad x^3 \geqslant 0; \qquad b_x = -1 \quad \text{otherwise} \tag{60}$$

$$b_x = +1 \quad \text{if} \quad x^1 \geqslant x^2 \qquad b_x = -1 \quad \text{otherwise} \tag{61}$$

By applying the results of Section 4, these boundary terms can be compared. For instance, in the cases in which the diagonal state (61) is translation invariant, then the states (60) and (61) should be also translation invariant.

We notice finally that such new extremal noninvariant states do appear in four dimensions. This is shown by the following proposition, which answers a problem proposed to us by D. Ruelle.

The boundary condition (61) which we shall denote $\Lambda(D)$, leads in four dimensions to a rigid diagonal interface. The proof of this fact is very similar to van Beijeren's,[5] with the following modifications: First we duplicate the system with respect to the diagonal plane $x^1 = x^2$ and second we observe that placing the spins $\sigma_x = +1$ for $x^1 = x^2 + 1$ and $\sigma_x = -1$ for $x^1 = x^2 - 1$ is the same as increasing the field K_x in the Hamiltonian (19). When this has been done one obtains a two-dimensional Ising system with $(+)$ boundary conditions in the diagonal plane. Therefore if x is a point in the diagonal plane $x^1 = x^2$, we obtain

$$\langle \sigma_x \rangle_{\Lambda(D)} \quad \text{(four dimensions)} \geqslant \langle \sigma_x \rangle^+ \quad \text{(two dimensions)}$$

This inequality can be compared to the van Beijeren's result, which for the four-dimensional case says

$$\langle \sigma_x \rangle^{\pm} \quad \text{(four dimensions)} \geqslant \langle \sigma_x \rangle^+ \quad \text{(three dimensions)}$$

for $x = 0$ and the $(\pm)$ boundary term. This suggests that the diagonal interface is less rigid than the horizontal interface. Hence a new temperature T_{R_D} may appear above which the diagonal interface is no longer rigid, and this temperature should be less than or equal to the conjectured temperature T_R (mentioned in Ref. 5) above which the horizontal rigid interface disappears.

The same argument as above shows that a rigid diagonal interface appears at low temperature for the model with next nearest neighbor interactions in three dimensions.

7. CONCLUDING REMARKS

Some comments on Section 3 have been given at the end of that section. They apply also to the results concerning estimates in Sections 4 and 5. We would like also to point out that in discussing Theorem 1 with Hegerfeld, he indicated that he has obtained a different proof of the decreasing properties (7) and (8). Hegerfeld's proof, which is also very short, is based on FKG inequalities.

Moreover, the results of Sections 4 and 5 could be extended to more general situations: some continuous spin systems or φ^4 Euclidean field theory if the results in Refs. 6, 7, and 22 were known in these cases.

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